Homenagem a Saponga

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Fixed points of nilpotent actions on surfaces of negative Euler characteristic

Javier Ribón

Framework

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We want to find *global fixed points*, i.e. common fixed points of all the diffeomorphisms in the group.

We also like *finite orbits*.

Cyclic theorems

Theorem

Let $f \in \mathrm{Homeo}_+(\mathbb{S}^2)$. Then $\mathrm{Fix}(f) \neq \emptyset$.

$L(f) = 2$

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Let $f \in \mathrm{Homeo}_+(S)$ where S is a compact surface of negative Euler characteristic. Then f has a finite orbit.

$L(f^n) \leq \chi(S) < 0$ for some n

$L(f) = \cdots = L(f^{2g+1}) = 0 \Rightarrow L(f^m) = 0$ $\forall m \in \mathbb{N}$

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Let $f \in \mathrm{Homeo}_+(S)$ where S is a compact surface of negative Euler characteristic. Then f has a finite orbit.

Moreover, if $f \in \mathrm{Homeo}_0(S)$ then f has a *contractible fixed point*.

Cyclic theorems

Theorem for $\chi(S) < 0$

If $f \in \mathrm{Homeo}_0(S)$ then f has a *contractible fixed point*.

$\mathrm{Homeo}_0(S) = \{f \in \mathrm{Homeo}(S) : f \text{ is isotopic to id}\}.$

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Let $\pi : \tilde{S} \to S$ be the universal covering of S.

Given $f \in \mathrm{Homeo}_0(S)$, there exists a unique $\tilde{f} \in \mathrm{Homeo}(S)$ such that $\pi \circ \tilde{f} = f \circ \pi$ and $\tilde{f}$ commutes with all deck transformations, the so called *identity lift* of f.

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We define the set of *contractible fixed points* of f as $\pi(\mathrm{Fix}(\tilde{f}))$.

Cyclic theorems

Theorem for $\chi(S) < 0$

If $f \in \mathrm{Homeo}_0(S)$ then f has a *contractible fixed point*.

Given a vector field X on S, the singular set of X is a subset of the set of contractible fixed points of its time-t map ϕ_t.

Cyclic theorems

Theorem (Brouwer)

Let $f \in \mathrm{Homeo}_+(\mathbb{R}^2)$ with $\mathrm{Fix}(f) = \emptyset$. Then we have $\lim_{n\to\infty} f^n(z) = \infty$ for any $z \in \mathbb{R}^2$.

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Corollary

Let $f \in \mathrm{Homeo}_+(\mathbb{R}^2)$ with an invariant compact set K. Then $\mathrm{Fix}(f) \neq \emptyset$.

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Let $f \in \mathrm{Homeo}_+(\mathbb{R}^2)$ with an invariant compact set K. Then $\mathrm{Fix}(f) \neq \emptyset$.

Corollary

Let $f \in \mathrm{Homeo}_+(\mathbb{R}^2)$ that has an invariant probability measure. Then $\mathrm{Fix}(f) \neq \emptyset$.

Lie algebras

Theorem (Lima '64)

Let $\mathfrak{g}$ be an abelian Lie algebra of vector fields on a surface of non-vanishing Euler characteristic. Then all vector fields in $\mathfrak{g}$ have a common singular point.

$$[X, Y] = 0$$

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where $Z(g) = \{X \in \mathfrak{g} : [X, Y] = 0 \; \forall Y \in \mathfrak{g}\}$ is the center of $\mathfrak{g}$.

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We say that $\mathfrak{g}$ is nilpotent if $\mathfrak{g}_k$ is trivial for some $k \in \mathbb{N}$.

Localization of fixed points

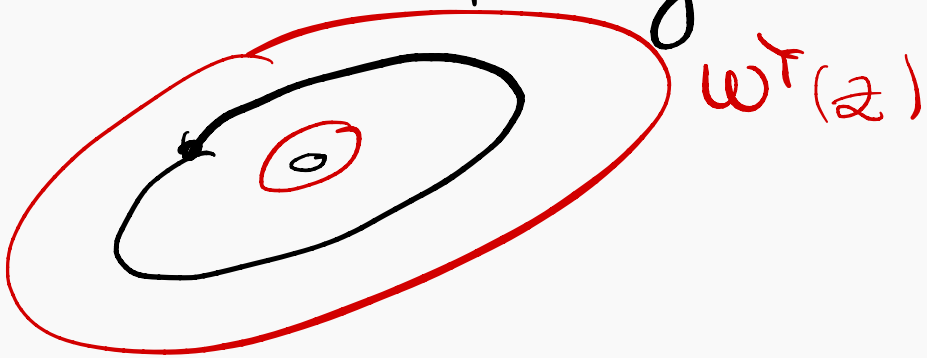
Theorem (Lima)

Let $X, Y \in \mathfrak{X}(\mathbb{S}^2)$ with $[X, Y] = 0$. Then X and Y have a common singular point.

$z \in \mathrm{Sing}(X)$, $\omega^+(z) \subset \mathrm{Sing}(X)$

You can assume $\omega^+(z) \cap \mathrm{Sing}(Y) = \emptyset$

$\omega^+(z)$ closed trajectory



Diffeomorphisms C^1 close to the identity

Theorem (Bonatti '89 and '90)

Let $f_1, \ldots, f_n \in \mathrm{Diff}^1(S)$ be commuting diffeomorphisms, C^1-close to the identity with $\chi(S) \neq 0$. Then they have a global fixed point.

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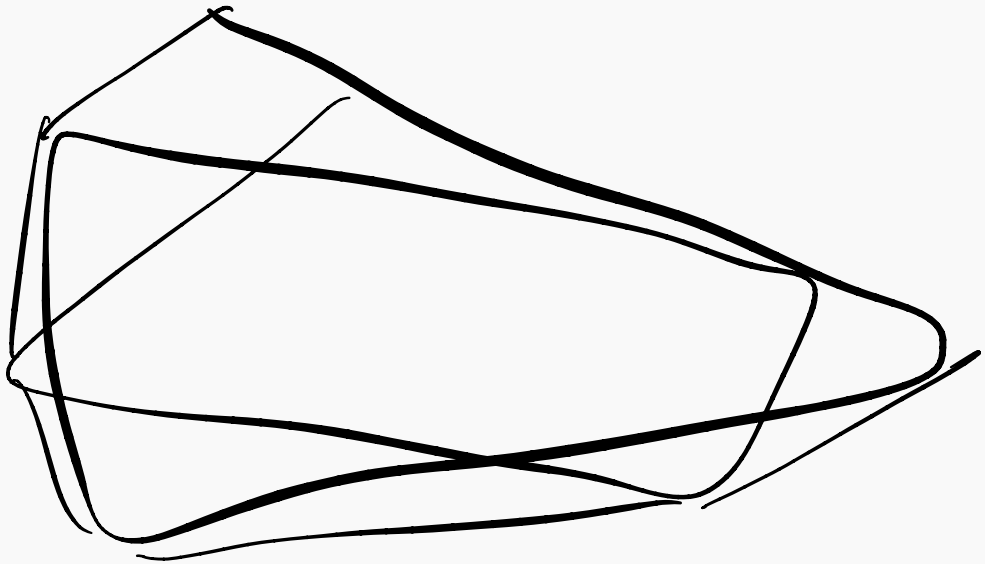
Theorem (Druck-Fang-Saponga '02)

Let $f_1, \ldots, f_n \in \mathrm{Diff}^1(\mathbb{S}^2)$ be generators of a nilpotent group G that are C^1-close to the identity. Then G has a global fixed point.

Localization

Theorem (Saponga '11)

Let $f_1, \ldots, f_n \in \mathrm{Diff}^1(\mathbb{R}^2)$ be commuting diffeomorphisms, C^1-close to the identity. Let $p \in \mathbb{R}^2$ whose orbit $\mathcal{O}_p(G)$ is bounded where $G = \langle f_1, \ldots, f_n \rangle$. Then G has a global fixed point in the convex hull of $\overline{\mathcal{O}_p(G)}$.



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Theorem (Ribón-Saponga-Velasco '14)

Let $f_1, \dots, f_n \in \mathrm{Diff}^1(\mathbb{R}^2)$ be generators of a nilpotent group that are C^1-close to the identity. Let $p \in \mathbb{R}^2$ whose orbit $\mathcal{O}_p(G)$ is bounded where $G = \langle f_1, \dots, f_n \rangle$. Then G has a global fixed point in the convex hull of $\overline{\mathcal{O}_p(G)}$.

Groups of C^1 diffeomorphisms

Theorem (Franks-Handel-Parwani '07)

Let G be an abelian subgroup of $\mathrm{Diff}^1(S)$ where S is a compact surface with $\chi(S) \neq 0$. Then G has a finite orbit.

If $\chi(S) < 0$ and $G < \mathrm{Diff}^1_0(S)$ then G has a global contractible fixed point.

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If $\chi(S) < 0$ and $G < \mathrm{Diff}^1_0(S)$ then G has a global contractible fixed point.

Theorem (Ribón '14 and '21)

Let G be a nilpotent subgroup of $\mathrm{Diff}^1(S)$ where S is a compact surface with $\chi(S) \neq 0$. Then G has a finite orbit whose cardinal is bounded by above by a function of $\chi(S)$.

If $\chi(S) < 0$ and $G < \mathrm{Diff}^1_0(S)$ then G has a global contractible fixed point.

The torus $\mathbb{T}^2$

Theorem (Ribón-Saponga '17)

Let G be a a nilpotent subgroup of $\mathrm{Diff}^1(\mathbb{T}^2)$. Suppose that there exists $f \in G$ such that $L(f) \neq 0$. Then G has a finite orbit.

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Theorem (Ribón-Saponga '19)

Let G be an irrotational nilpotent subgroup of $\mathrm{Diff}^1_0(\mathbb{T}^2)$. Then G has a global fixed point.

Main result

Theorem for $\chi(S) < 0$

Let G be a nilpotent subgroup of $\mathrm{Diff}^1(S)$. Then G has a finite orbit whose cardinal is bounded by above by a function of $\chi(S)$.

If $G < \mathrm{Diff}_0^1(S)$ then G has a global contractible fixed point.

G

$Z(G) = \{ f \in G ; fg = gf \ \forall g \in G \}$

$G_1 = \frac{G}{Z(G)}$, $G_2 = \frac{G_1}{Z(G_1)}$, ...

G is nilpotent if $G_k = \{1\}$ for some k

Cuilpctant

$H \trianglelefteq G \Longrightarrow H \cap Z(G) \neq \{1\}$

Irreducible case

- Minimality of fixed points for pseudo-Anosov or finite orbit homeomorphisms.
- Existence of models for nilpotent subgroups of $\mathrm{MCG}(S)$ consisting of irreducible elements.

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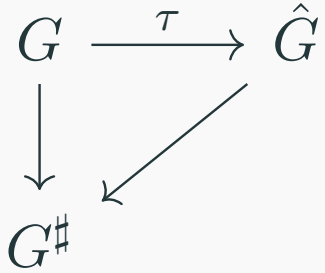
Consider $G < \mathrm{Homeo}(S)$ whose projection $G^\sharp$ on $\mathrm{MCG}(S)$ consists on irreducible classes. The group $G^\sharp$ is virtually cyclic.

$\exists\, \hat{G} < \mathrm{Homeo}(S)$ consisting of finite order and pseudo-Anosov homeomorphisms and a morphism of groups $\tau : G \to \hat{G}$ such that



commutes. The group $\hat{G}$ is virtually cyclic.

From finite orbits to global fixed points

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$$H \lhd G < \mathrm{Diff}_0^1(S),\ |G : H| < \infty \text{ and } \mathrm{Fix}_c(H) \neq \emptyset$$
$$\Downarrow$$
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Theorem (Franks, Handel, Parwani, Le Calvez, Ribón, Saponga)

Let G be a nilpotent subgroup of $\mathrm{Diff}_+^1(\mathbb{R}^2)$. Assume that there exists a compact invariant set. Then $\mathrm{Fix}(G) \neq \emptyset$.

$\mathrm{Fix}_c(H) \neq \emptyset \Rightarrow \mathrm{Fix}(\widetilde{H}) \neq \emptyset$

$\widetilde{H} \lhd \widetilde{G}$, $|\widetilde{G} : \widetilde{H}| = |G : H|$, $\widetilde{G}$ has a finite orbit

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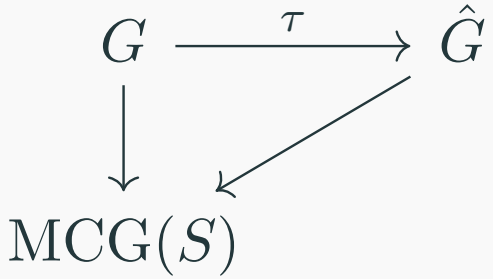
From $\hat{G}$ to G



Setting: $\pi : \tilde{S} \to S$. Let $z \in \mathrm{Fix}(\hat{G})$. Fix $\tilde{z} \in \pi^{-1}(z)$.

Given $f \in G$, $\exists$ an isotopy $(f_t)_{t\in[0,1]}$ with $f_0 = \tau(f)$ and $f_1 = f$.

$\tilde{f}_0$ is the lift of f_0 to $\tilde{S}$ such that $\tilde{f}_0(\tilde{z}) = \tilde{z}$.

We consider the lift $(\tilde{f}_t)_{t\in[0,1]}$ of $(f_t)_{t\in[0,1]}$.

We define $\tilde{f} = \tilde{f}_1$.

$\tilde{G} := \{\tilde{f} : f \in G\}$ is a nilpotent subgroup of $\mathrm{Diff}_0^1(\tilde{S})$.

If $\hat{G} \neq \{1\}$ then $\exists$ a compact $\tilde{G}$-invariant set.

From $\hat{G}$ to G

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$$\Downarrow$$

$$\mathrm{Fix}(\tilde{G}) \neq \emptyset$$

$$\Downarrow$$

$$\mathrm{Fix}_c(G) \neq \emptyset.$$

Irreducible case

$$\tau : G \to \hat{G}$$

If $\hat{G} \neq \{1\}$ then $\exists$ a compact $\tilde{G}$-invariant set.

We want to show $G < \mathrm{Diff}_0^1(S) \implies \mathrm{Fix}_c(G) \neq \emptyset$.

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These G-invariant sets are typically fixed point sets of normal subgroups of G.

They can be infinite. This is where the C^1 hypothesis is used.

Every $f \in \mathrm{Diff}^1(S)$ is locally isotopically trivial relative to $\mathrm{Fix}(f)$.

S \ K

Thurston-Nielsen normal form

Let G be a finitely generated nilpotent subgroup of $\mathrm{Diff}_+(S)$ where S is a finitely punctured surface with $\chi(S) < 0$. Then there exist a finite family $\mathcal{R}$ of pairwise disjoint, pairwise non-homotopic simple closed curves in M and a family of pairwise disjoint open annuli $\mathcal{A}$ in M of the same cardinal as the family $\mathcal{R}$ such that every curve in $\mathcal{R}$ is a core curve of exactly an annulus in $\mathcal{A}$ and given $\phi \in G$ there exists $\theta_\phi \in \mathrm{Homeo}_+(S)$ such that ϕ is isotopic to θ_ϕ relative to K and $\theta_\phi(\mathcal{A}) = \mathcal{A}$. Moreover given a connected component X of $S \setminus \mathcal{A}$ we have $\chi(X) < 0$ and

- the set $\{(\theta_\phi)_{|X} : \phi \in J\}$, where $J \subset G$ is the stabilizer of X modulo isotopy, is a group of pseudo-Anosov and finite order elements that is virtually cyclic and nilpotent.

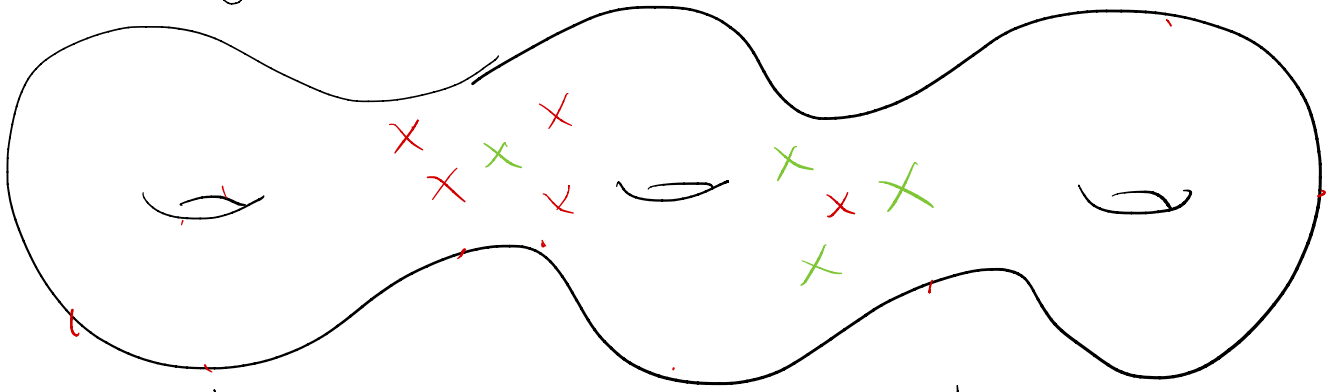
Outline of the proof

Theorem

Let G be a nilpotent subgroup of $\mathrm{Diff}_0^1(S)$ where S is a compact surface with $\chi(S) < 0$. Then G has a global contractible fixed point.

Irreducible case

$G = \langle f, g \rangle$; $K = \mathrm{Fix}_c(f) \cup \mathrm{Fix}_c(g)$

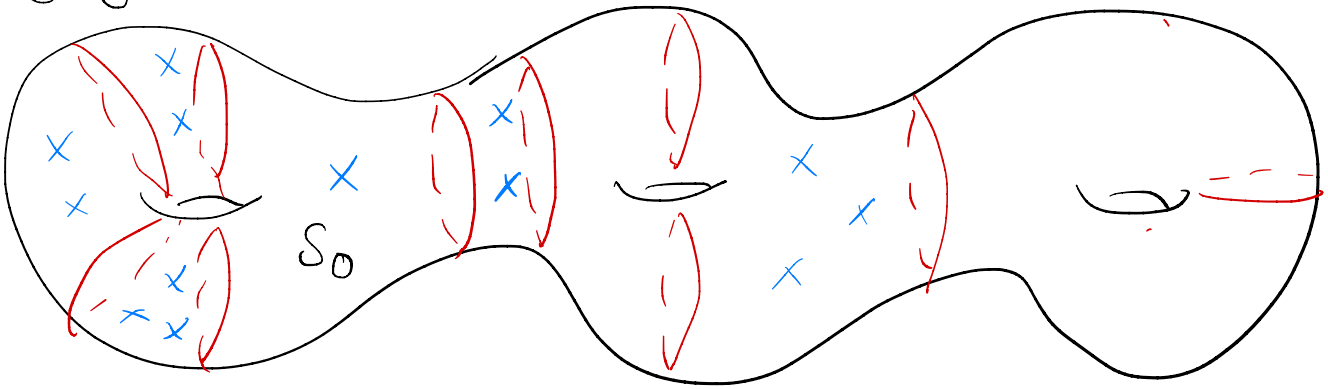


Assume G^H is irreducible $\Rightarrow$ G^H is virtually cyclic

$\exists\, H \lhd G$, $|G:H| < \infty \Rightarrow \mathrm{Fix}_c(f) \cup \mathrm{Fix}_c(g) \subset \mathrm{Fix}_c(H)$

Hence $\mathrm{Fix}_c(G) \neq \emptyset$

$G = \langle f, g \rangle$; $K = \mathrm{Fix}_c(f) \cup \mathrm{Fix}_c(g) \cup \mathrm{Fix}_c(fg) \cup \mathrm{Fix}_c(f^2 g) \cup \dots$



$f \sim f_0$ rel K where $(f_0)_{|S_0} \equiv id_{S_0}$

$\tilde{S}_0$ = connected component of $\pi^{-1}(S_0)$ in $\tilde{S}$

$\tilde{f}_0$ = lift of f_0 to $S_0 \setminus (K \setminus S_0)$ where $(\tilde{f}_0)_{|\tilde{S}_0} \equiv id$

$\tilde{f} = \tilde{f}_1$ where $(\tilde{f}_t)$ is the lift to $S_0 \setminus (K \setminus S_0)$

Outline of the proof

Theorem

Let G be a nilpotent subgroup of $\mathrm{Diff}_0^1(S)$ where S is a compact surface with $\chi(S) < 0$. Then G has a global contractible fixed point.

- We consider isotopy classes relative to a compact G-invariant set.
- The irreducible case is simple.
- We can reduce the setting to the irreducible case.
  We localize fixed point sets.

Thank you